Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-04-06

Conceptual Review

Question 1. Let A be the portion of the unit circle in the first quadrant, and let C be the entire unit circle. Without doing any calculations, can you determine whether each of the following quantities is positive, negative, or zero?

(a) $\int_A xy\,\mathrm{d}s$
(b) $\int_C xy\,\mathrm{d}s$
(c) $\int_A x^2 - y^2\,\mathrm{d}s$

Question 2. Consider

- $\mathbf{r}_1(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$,
- $\mathbf{r}_2(t) = \langle t^2, t^4 \rangle$ for $0 \le t \le 1$, and
- $\mathbf{r}_3(t) = \langle -t, t^2 \rangle$ for $-1 \le t \le 0$.

These are three different parametrizations of the same curve C in the xy-plane. Let $f(x, y)$ be some scalar function, and $\mathbf{F}(x, y)$ some vector field.

Would these different parametrizations give the same answer for $\int_C f(x, y)\,\mathrm{d}s$? What about for $\int_C \mathbf{F}(x, y) \cdot \mathrm{d}\mathbf{r}$?

Question 3. Let C be a curve defined by $f(x, y) = 0$, starting at some point (x_0, y_0) and ending at (x_1, y_1). What is the value of $\int_C \nabla f \cdot \mathrm{d}\mathbf{r}$?

Problems

Problem 1. Show that the vector field $\langle 1 + x^2, xy \rangle$ is tangent to the hyperbola $y^2 - x^2 = 1$ at all points on the hyperbola.

Problem 2.

(a) Find a function $f(x, y)$ such that $\nabla f(x, y) = \langle (1 + xy)e^{xy}, x^2 e^{xy} \rangle$.

The next part actually requires the Fundamental Theorem of Line Integrals, which I overlooked while preparing this worksheet. We will discuss it later.

(b) Let C be the unit circle $x^2 + y^2 = 1$ oriented counterclockwise. Compute

$$\int_C (1 + xy)e^{xy}\,\mathrm{d}x + (x^2 e^{xy} + x)\,\mathrm{d}y.$$

Here are some brief answers or comments on the exercises. As always, I am willing to elaborate further on request.

Question 1.

(a) Positive because the integrand is.
(b) Zero. The curve C is symmetric under the change of variables $y \mapsto -y$ (i.e. reflection across the x-axis). This means $\int_C xy\,\mathrm{d}s = \int_C x(-y)\,\mathrm{d}s$, which can only be possible if the integral is equal to zero.
(c) Also zero. Use the symmetry which interchanges x and y.

Question 2. All three would give the same answer for the scalar line integral, whereas the first two would give the negative of the answer of the third for the vector line integral (because the parametrization is going the opposite way).

Question 3. At all points along the curve, ∇f and $\mathbf{T}$ (or $\mathbf{r}'(t)$) are orthogonal. So the integrand is always zero, meaning the integral will be equal to zero as well.

Problem 1. Write $f(x,y) = y^2 - x^2 - 1$. Then the gradient vector field $\nabla f = \langle -2x, 2y\rangle$ is perpendicular to the hyperbola. So it suffices to check that the given vector field is always orthogonal to the gradient vector field, which we can verify using the dot product:

$$\langle 1+x^2, xy\rangle \cdot \langle -2x, 2y\rangle = -2x - 2x^3 + 2xy^2.$$

Note moreover that we only care about points on the hyperbola, i.e. we have $y^2 - x^2 = 1$. So the above expression further simplifies to

$$-2x - 2x^3 + 2x(1+x^2) = 0$$

as desired.

Problem 2. Just the first part:

$$f(x,y) = xe^{xy} + 17.$$

Of course the +17 is totally arbitrary.